Further Pure Math 2 (FP2): HW 1 — Full name: Answer Keys
Dr.Zhongmin Jin, Biweek, Week1. — This Homework is worth a total of ~~32~~ **points**.

1. The curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$ intercept the point where $x = a$. $\boxed{4}$

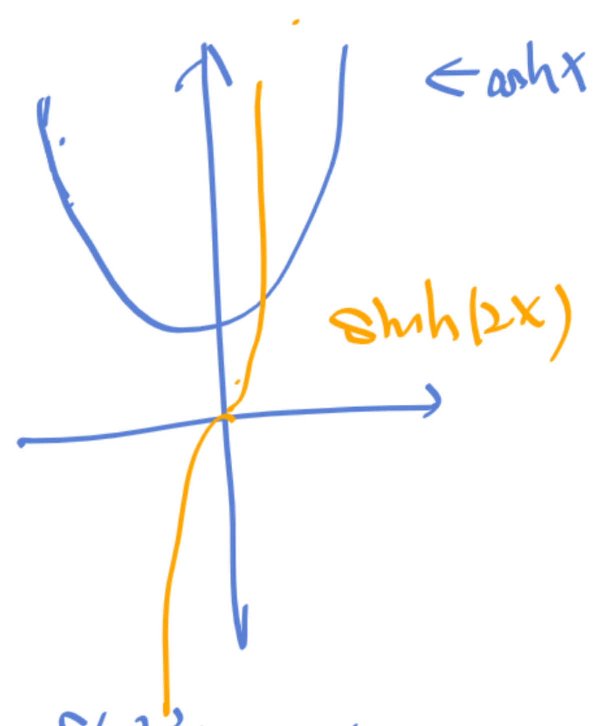
   (a) Find the exact value of a, giving your answer in logarithmic form.
   (b) Sketch C_1 and C_2 on the same diagram.
   (c) Find the exact vlaue of the length of the arc of C_1 from 0 to $x = a$. (the arc length formula is given as $\int_a^b \sqrt{1 + (y')^2} dx$)

Solution: (a) We have eq: $\cosh(x) = \sinh(2x)$
$= 2\sinh(x)\cosh(x)$

Hence $\sinh(x) = \frac{1}{2}$, since $\cosh(x) \geq 1 > 0$

$x = \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})\Big|_{x=\frac{1}{2}} = \ln\left(\frac{1}{2} + \sqrt{\frac{5}{4}}\right) = \ln\left(\frac{1+\sqrt{5}}{2}\right)$

(b)



(c) By arc length formula, we have (we have $\cosh^2 x - \sinh^2 x = 1$)

$$S = \int_0^a \sqrt{1 + (\sinh x)^2} = \int_0^a \cosh(x)\, dx$$
$$= \sinh(x)\Big|_0^a = \sinh(a) = \frac{1}{2}$$

2. (a) Starting from the definitions of tanh and sech in terms of exponential, prove that $\boxed{4}$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta \tag{1}$$

   (b) The variables x and y are such that $\tanh y = \cos(x + \frac{1}{4}\pi)$. By differentiating the equation with respect to x, show that

$$\frac{dy}{dx} = -\csc(x + \frac{1}{4}\pi) \tag{2}$$

(a) proof: $1 - \tanh^2\theta = 1 - \dfrac{\left(\frac{e^\theta - e^{-\theta}}{2}\right)^2}{\left(\frac{e^\theta + e^{-\theta}}{2}\right)^2} = 1 - \dfrac{e^{2\theta} + e^{-2\theta} - 2}{e^{2\theta} + e^{-2\theta} + 2}$

$= \dfrac{4}{e^{2\theta} + e^{-2\theta} + 2} = \dfrac{1}{\left(\frac{e^\theta + e^{-\theta}}{2}\right)^2} = \dfrac{1}{\cosh^2\theta} = \operatorname{sech}^2\theta$ □

(b) $(\tanh x)' = \left(\dfrac{\sinh x}{\cosh x}\right)' = \dfrac{1}{\cosh^2(x)} = \operatorname{sech}^2(x)$

Hence, $\operatorname{sech}^2(y)\dfrac{dy}{dx} = -\sin(x + \frac{1}{4}\pi)$. ($\operatorname{sech}^2(y) = 1 - \tanh^2(y) = 1 - \cos^2(x + \frac{1}{4}\pi)$)

$\Leftrightarrow (1 - \cos^2(x + \frac{1}{4}\pi))\dfrac{dy}{dx} = -\sin(x + \frac{1}{4}\pi)$

$\sin^2(x + \frac{1}{4}\pi)\dfrac{dy}{dx} = -\sin(x + \frac{1}{4}\pi) \Leftrightarrow \dfrac{dy}{dx} = -\csc(x + \frac{1}{4}\pi)$.

3. (a) Sketch the graph of $y = \coth x$ for $x > 0$ and state the equation of the asymptotic. $\boxed{6}$

(b) Starting from the definition of coth and csch in terms of exponential, prove that

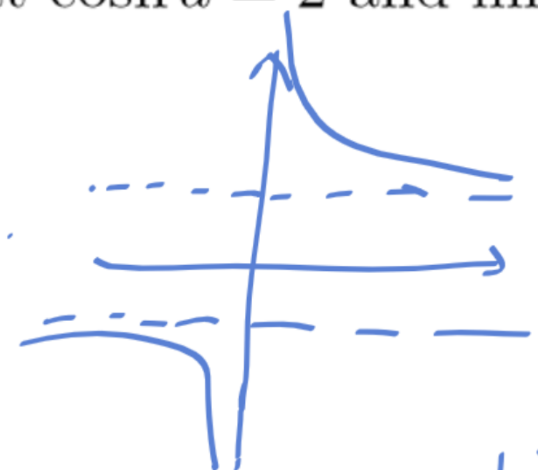
$$\coth^2 x - \operatorname{csch}^2 x = 1. \qquad (3)$$

The curve C has equation $y = \ln(\coth \frac{x}{2})$ for $x > 0$.

(c) Show that $\frac{dy}{dx} = -\operatorname{csch} x$.

(d) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where a is a positive constant. Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a.

Solutions (a)



horizontal asymptote $y = \pm 1$

(b) $\coth^2 x - \operatorname{csch}^2 x = \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2 = \frac{e^{2x}+e^{-2x}+2-4}{(e^x-e^{-x})^2}$

$= \frac{e^{2x}+e^{-2x}-2}{(e^x-e^{-x})^2} = \frac{(e^x-e^{-x})^2}{(e^x-e^{-x})^2} = 1$

(c) $\frac{dy}{dx} = \frac{1}{\coth(\frac{x}{2})}\left(\coth\frac{x}{2}\right)' = \frac{1}{\coth(\frac{x}{2})} \cdot \frac{-1}{\sinh^2(\frac{x}{2})} \cdot \frac{1}{2} = \frac{-1}{\sinh^2(x)}$

$= \frac{\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})} \cdot \frac{-1}{\sinh^2(\frac{x}{2})} \cdot \frac{1}{2} = \frac{-1}{2\sinh(\frac{x}{2})\cosh(\frac{x}{2})} = -\frac{1}{\sinh x} = -\operatorname{csch}(x)$

(d) Arc length $= \int_a^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^{2a} \sqrt{1+(-\operatorname{csch}(x))^2}\,dx$

$= \int_a^{2a} \sqrt{1+\frac{1}{\sinh^2(x)}}\,dx = \int_a^{2a} \sqrt{\frac{\sinh^2(x)+1}{\sinh^2(x)}}\,dx = \int_a^{2a} \sqrt{\coth^2(x)}\,dx$

$= \int_a^{2a} \coth(x)\,dx = \int_a^{2a} \frac{\cosh(x)}{\sinh(x)}\,dx = \ln(\sinh(x))\Big|_a^{2a}$

$= \ln\left(\frac{\sinh(2a)}{\sinh(a)}\right) = 4$

$\therefore \frac{\sinh(2a)}{\sinh(a)} = 4$

$\Leftrightarrow \frac{2\sinh(a)\cosh(a)}{\sinh(a)} = 4$

$\Leftrightarrow \cosh(a) = 2$

$\Leftrightarrow a = \pm\ln(2+\sqrt{2^2-1})$

$= \pm\ln(2+\sqrt{3})$

4. Solve $17\sinh x + 16\cosh x = 8$. $\boxed{6}$

Solutions:

$17\,\frac{e^x-e^{-x}}{2} + 16\,\frac{e^x+e^{-x}}{2} = 8$

$\frac{1}{2}(33e^x - e^{-x}) = 8$

$33e^x - 16 - e^{-x} = 0$

$e^{-x}(33e^{2x} - 16e^x - 1) = 0$

$e^x = \frac{16+\sqrt{16^2+4\cdot 33}}{2\cdot 33} = \frac{8+\sqrt{97}}{33} \Rightarrow x = \ln\left(\frac{8+\sqrt{97}}{33}\right) \simeq 0.541$ (3 s.f.)

5. Prove the identity $\sinh 3x = 3\sinh x + 4\sinh^3 x$. $\boxed{6}$

proof: $\sinh 3x = \sinh(x+2x) = \sinh(x)\cosh(2x) + \cosh(x)\sinh(2x)$

$= \sinh(x)\left(\cosh^2(x)+\sinh^2(x)\right) + \cosh(x)\,2\sinh(x)\cosh(x)$

$= \sinh(x)\cosh^2(x) + \sinh^3(x) + 2\cosh^2(x)\sinh(x)$

$= \sinh^3(x) + 3\sinh(x)\cosh^2(x)$

$= \sinh^3(x) + 3\sinh(x)\left(1+\sinh^2(x)\right)$

$= 3\sinh(x) + 4\sinh^3(x)$ $\square$

6. Solve $\tanh^2 x + 5\,\mathrm{sech}\, x - 5 = 0$ in logarithms. $\boxed{6}$

Solution: $\tanh^2 x - 1 = \dfrac{\sinh^2(x)}{\cosh^2(x)} - 1 = \dfrac{\sinh^2(x)-\cosh^2(x)}{\cosh^2(x)}$

$\therefore \quad \mathrm{sech}^2(x) = 1 - \tanh^2(x) \qquad = \dfrac{-1}{\cosh^2(x)} = -\mathrm{sech}^2(x)$

$1 - \mathrm{sech}^2(x) + 5\,\mathrm{sech}(x) - 5 = 0$

$\mathrm{sech}^2(x) - 5\,\mathrm{sech}(x) + 4 = 0$

$\therefore \quad \mathrm{sech}(x) = 1 \quad , \quad \mathrm{sech}(x) = 4$

$\cosh(x) = 1 \quad , \quad \cosh(x) = \frac{1}{4}$

no root

$x = 0$